

A Fourfold Refined Enumeration of Alternating Sign Trapezoids and Column Strict Shifted Plane Partitions

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Alternating Sign Trapezoids

Definition (Ayyer; Aigner; Behrend, Fischer)

An (n, l) -alternating sign trapezoid (AST) (for $l \geq 2$) is a trapezoidal array of integers with n rows, l entries in the bottom row and entries -1 , 0 or $+1$ such that

- the sum of the entries in each row equals 1,
- the nonzero entries alternate in sign along each row and each column,
- the top-most nonzero entry in each column is 1,
- the entries in the central $l - 2$ columns sum to 0.

0	0	0	1	0	0	0	0	0	0	0	0
	1	0	-1	0	0	0	0	0	1	0	
		0	1	0	0	0	0	0	0		
			0	0	0	1	0	0			
				1	0	-1	1				

(5, 4)-AST

Statistics on ASTs

$$\begin{array}{cccccccccccc} 0 & 0 & 0 & 1 & 0 & \boxed{0} & \boxed{0} & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 & 0 & \boxed{0} & \boxed{0} & 0 & 0 & 1 & 0 & \\ & & 0 & 1 & 0 & \boxed{0} & \boxed{0} & 0 & 0 & 0 & & \\ & & & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & & & \\ & & & & 1 & \boxed{0} & \boxed{-1} & 1 & & & & \end{array}$$

1-column vector: $\mathbf{c} = (-4, -2, -1, 1, 3)$, length n

1-columns on the left: $r = 3$

10-columns on the left: $s = 1$

10-columns on the right: $t = 1$

-1s: $q = 2$

→ weight: $R^3 S^1 T^1 Q^2$

Statistics on ASTs

-5	-4	-3	-2	-1			1	2	3	4	5
0	0	0	1	0	0	0	0	0	0	0	0
	1	0	-1	0	0	0	0	0	1	0	
		0	1	0	0	0	0	0	0		
			0	0	0	1	0	0			
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		0	1	0	0	0	0	0	0			
			0	0	0	1	0	0				
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		0	1	0	0	0	0	0	0			
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Column Strict Shifted Plane Partitions

Definition (Andrews 1979, Mills, Robbins, Rumsey 1987)

A **column strict shifted plane partition (CSSPP)** is filling of a shifted Young diagram with positive integers such that the entries decrease along each row and strictly decrease down each column. It is of **class k** if the first entry of each row i is exactly k plus its row length.

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class 4

Observation

CSSPPs of class 2 with no row length larger than n



descending plane partitions with no part larger than $n - 1$

Statistics on CSSPPs (of class k)

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class $k = 4$

For a fixed parameter $d \in \{1, \dots, k\}$:

rows: $r = 3$

parts $\pi_{i,j}$ equal to $j - i + d$: $s = 1$

1s: $t = 1$

parts $\pi_{i,j}$ less or equal to $j - i + k$ but not 1 or $j - i + d$: $q = 2$

→ weight: $R^4 S^1 T^1 Q^2$

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	7	7	5	
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3	4	5	6	7
	3	4	5	
		3	4	

filling with $j - i + d$, $d = 3$

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Statistics on CSSPPs (of class k)

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class $k = 4$

4	5	6	7	8
	4	5	6	
		4	5	

filling with $j - i + k$, $k = 4$

For a fixed parameter $d \in \{1, \dots, k\}$:

rows: $r = 3$

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1s: $t = 1$

parts $\pi_{i,j}$ less or equal to $j - i + k$ but not 1 or $j - i + d$: $q = 2$

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(Joint) distributions

Theorem (Behrend, Di Francesco, Zinn-Justin 2012 + Ayer, Behrend, Fischer 2016)

The distribution of the Q -statistic on CSSPPs of class 2 with no row length larger than n and on $(n, 3)$ -ASTs coincide.

Theorem (Fischer 2018)

The joint distribution of the R -, S -, and T -statistics on CSSPPs of class $l - 1$ with no row length larger than n and on (n, l) -ASTs coincide.

Theorem (H.)

The joint distribution of the Q -, R -, S -, and T -statistics on CSSPPs of class $l - 1$ with no row length larger than n and on (n, l) -ASTs coincide.

An example: $n = 2, l = 4$

(2, 4)-ASTs:

1 0 0 0 0 0	0 0 0 0 1 0	0 0 0 0 0 1	1 0 0 0 0 0
1 0 0 0	1 0 0 0	1 0 0 0	0 0 0 1
R^2	RT	R	R
0 1 0 0 0 0	0 0 0 0 0 1	0 0 1 0 0 0	0 0 0 1 0 0
0 0 0 1	0 0 0 1	1 -1 0 1	1 0 -1 1
RS	1	QR	QR

CSSPPs of class 3 with at most 2 parts in the first row:

\emptyset	4	5 5	5 4	5 3	5 2	5 1	5 5	4
$d = 1:$	1	R	R	QR	QR	RS	RT	R^2
$d = 2:$	1	R	R	QR	RS	QR	RT	R^2
$d = 3:$	1	R	R	RS	QR	QR	RT	R^2

The proof

Generating function for ASTs:

- ASTs \longleftrightarrow Trees
- Operator formula for ASTs with given 1-column vector
- Constant term expression
- Summation over all possible 1-column vectors

Generating function for CSSPPs:

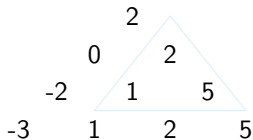
- CSSPPs \longleftrightarrow Family of nonintersecting lattice paths
- Lindström-Gessel-Viennot lemma

Monotone Triangles

Definition

A **monotone triangle (MT)** of order n is a triangular array of integers with n rows such that the entries

- strictly increase along rows,
- weakly increase along \nearrow -diagonals, and
- weakly increase along \searrow -diagonals.

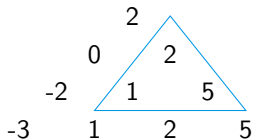


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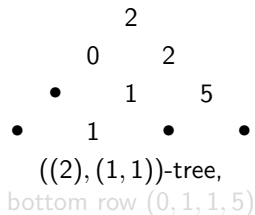
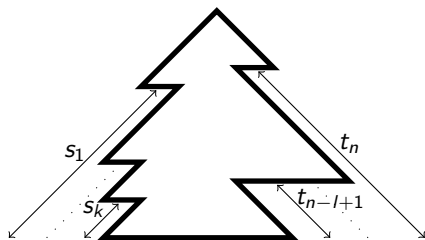
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Trees

Definition

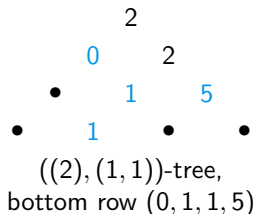
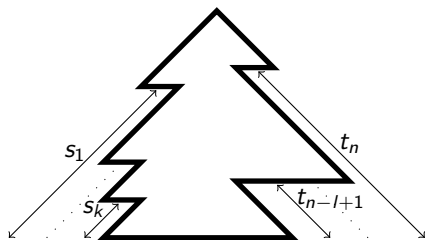
For a weakly decreasing sequence $\mathbf{s} = (s_1, \dots, s_k)$ and a weakly increasing sequence $\mathbf{t} = (t_{n-l+1}, \dots, t_n)$ of nonnegative integers ($k + l \leq n$), a (\mathbf{s}, \mathbf{t}) -tree is a MT of order n with truncated diagonals: the s_i bottom entries of the i th \nearrow -diagonal and the t_j bottom entries of the j th \searrow -diagonal are deleted.



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ASTs and Trees

```
0 0 0 1 0 0 0 0 0 0 0 0
  1 0 -1 0 0 0 0 0 0 1 0
    0 1 0 0 0 0 0 0 0
      0 0 0 1 0 0
        1 0 -1 1
```

- Add 0s for a rectangular shape
- Replace entries by partial column sums
- Record positions of 1s in shape of a MT
- Remove additional entries

ASTs and Trees

```
0 0 0 1 0 0 0 0 0 0 0 0
0 1 0 -1 0 0 0 0 0 1 0 0
0 0 0 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 0 0 0 0 0
0 0 0 0 1 0 -1 1 0 0 0 0
```

- Add 0s for a rectangular shape
- Replace entries by partial column sums
- Record positions of 1s in shape of a MT
- Remove additional entries

ASTs and Trees

0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	0	0
0	1	0	1	0	0	0	0	0	1	0	0
0	1	0	1	0	0	1	0	0	1	0	0
0	1	0	1	1	0	0	1	0	1	0	0

- Add 0s for a rectangular shape
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ASTs and Trees

$$\begin{array}{cccccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\ & & & & 1 & 0 & -1 & 1 & & & & \end{array}$$

(5,4)-AST

\longleftrightarrow

$$\begin{array}{ccc} & & -2 \\ -4 & & 4 \\ & -2 & 4 \\ -2 & & 1 \\ & -1 & 2 \end{array}$$

tree with 5 rows

- *R*-weight
- 1-column vector *c*
- *Q*-weight
- *S*-weight/*T*-weight

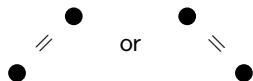
ASTs and Trees

$$\begin{array}{cccccccccccc} & & -2 & & & & & & 4 & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & \\ & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & 0 & 0 & 0 & 1 & 0 & 0 & & & & & \\ & & & 1 & 0 & -1 & 1 & & & & & & \end{array}$$

\longleftrightarrow

$$\begin{array}{ccc} & -2 & \\ -4 & 4 & \\ & -2 & 4 \\ -2 & 1 & \\ & -1 & 2 \end{array}$$

- R -weight
- 1-column vector \mathbf{c}
- Q -weight
- S -weight/ T -weight



at the bottom row

Main Tool: Operator Formulae

Main tool for enumeration:

- **shift operator**: $E_x f(x) := f(x + 1)$
- **forward difference operator**: $\Delta_x := E_x - \text{id}_x$
- **backward difference operator**: $\delta_x := \text{id}_x - E_x^{-1}$

Abbreviation: $E_k f(k) := E_x f(x)|_{x=k}$

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Abbreviation: $E_k f(k) := E_x f(x)|_{x=k}$

Enumeration of MTs

Theorem (Fischer 2006)

The number $\mathbf{MT}_n(k_1, \dots, k_n)$ of MTs of order n with bottom row (k_1, \dots, k_n) is given by

$$\prod_{1 \leq s < t \leq n} (E_{k_s} + E_{k_t}^{-1} - E_{k_s} E_{k_t}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.$$

Theorem (Fischer 2010)

The Q -enumeration ${}^Q \mathbf{MT}_n(k_1, \dots, k_n)$ of MTs of order n with bottom row (k_1, \dots, k_n) is given by

$$\prod_{1 \leq s < t \leq n} (E_{k_s} + E_{k_t}^{-1} - (2 - Q) E_{k_s} E_{k_t}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.$$

Enumeration of Trees

Observation

The difference operators $-\Delta_x = \text{id}_x - E_x$ and $\delta_x = \text{id}_x - E_x^{-1}$ cut off entries!

Theorem (Fischer 2011)

The number of (\mathbf{s}, \mathbf{t}) -trees with n rows and bottom row (k_1, \dots, k_n) is given by

$$\prod_{i=1}^k (-\Delta_{k_i})^{s_i} \prod_{j=n-l+1}^n (\delta_{k_j})^{t_j} \mathbf{MT}_n(k_1, \dots, k_n).$$

Enumeration of Trees

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Enumeration of Trees

Theorem (Fischer 2018)

The Q -enumeration of (\mathbf{s}, \mathbf{t}) -trees with n rows and bottom row (k_1, \dots, k_n) is given by

$$\prod_{i=1}^k \left(-{}^Q \Delta_{k_i}\right)^{s_i} \prod_{j=n-l+1}^n \left({}^Q \delta_{k_j}\right)^{t_j} {}^Q \mathbf{MT}_n(k_1, \dots, k_n),$$

where

$${}^Q \Delta_x := (Q - (1 - Q) \Delta_x)^{-1} \Delta_x,$$

$${}^Q \delta_x := (Q - (Q - 1) \delta_x)^{-1} \delta_x.$$

Constant term expression

Define the (anti)symmetriser of a formal Laurent series:

$$\mathbf{Sym}_{x_1, \dots, x_m} f(x_1, \dots, x_m) := \sum_{\sigma \in \mathfrak{S}_m} f(x_{\sigma(1)}, \dots, x_{\sigma(m)})$$

$$\mathbf{ASym}_{x_1, \dots, x_m} f(x_1, \dots, x_m) := \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(m)})$$

Theorem (Fischer, Riegler 2015)

^Q $\mathbf{MT}_n(x_1, \dots, x_n)$ is equivalent to

$$\mathbf{CT}_Y \mathbf{Sym}_Y \left(\prod_{i=1}^n (1 + Y_i)^{x_i} \prod_{1 \leq i < j \leq n} (Q - (1 - Q)Y_i + Y_j + Y_i Y_j) \right. \\ \left. \times \prod_{1 \leq i < j \leq n} (Y_j - Y_i)^{-1} \right).$$

ASTs with given 1-column vector

Theorem (H.)

The Q -enumeration of (n, l) -ASTs with 1-columns in positions $\mathbf{c} = (c_1, \dots, c_n)$ is given by

$$\prod_{i=1}^m \left(-{}^Q \Delta_{\tilde{c}_i}\right)^{-c_i-1} \prod_{j=m+1}^n {}^Q \delta_{\tilde{c}_j}^{c_j-1} {}^Q \mathbf{MT}_n(\tilde{c}_1, \dots, \tilde{c}_n),$$

where $\tilde{\mathbf{c}} = (c_1, \dots, c_m, c_{m+1} + l - 3, \dots, c_n + l - 3)$.

To incorporate S and T , add additional difference operators:

$$\prod_{i=1}^m \left(\text{id} - \frac{S}{Q} \delta_{\tilde{c}_i}\right) \left(\text{id} + {}^Q \Delta_{\tilde{c}_i}\right) \left(-{}^Q \Delta_{\tilde{c}_i}\right)^{-c_i-1} \\ \times \prod_{i=m+1}^n \left(\text{id} + \frac{T}{Q} \Delta_{\tilde{c}_i}\right) \left(\text{id} - {}^Q \delta_{\tilde{c}_i}\right) {}^Q \delta_{\tilde{c}_i}^{c_i-1} {}^Q \mathbf{MT}_n(\tilde{c}_1, \dots, \tilde{c}_n).$$

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Constant term expression

This can be transformed into the constant term of

$$\begin{aligned} \mathbf{Sym}_Y & \left(\prod_{i=1}^m (-Y_i)^{-c_i-1} (1+Y_i)^{c_i} (Q - (1-Q)Y_i)^{c_i+1} \frac{Q - (S-Q)Y_i}{Q - (1-Q)Y_i} \right. \\ & \times \prod_{i=m+1}^n Y_i^{c_i-1} (1+Y_i)^{c_i+l-3} (Q+Y_i)^{-c_i+1} \frac{Q+TY_i}{Q+Y_i} \\ & \left. \times \prod_{1 \leq i < j \leq n} (Q - (1-Q)Y_i + Y_j + Y_i Y_j) \prod_{1 \leq i < j \leq n} (Y_j - Y_i)^{-1} \right). \end{aligned}$$

Sum over 1-columns

Remaining step: Sum over all possible 1-column vectors.

- First, sum over all $-n \leq c_1 < \dots < c_m < 0 < c_{m+1} < \dots < c_n \leq n$.
- Then, sum over all $0 \leq m \leq n$.

$$\begin{aligned} \text{Sym}_{\mathbf{Y}} & \left(\sum_{c_1 < \dots < c_m < 0} \left(\prod_{i=1}^m (-Y_i)^{-c_i-1} (1+Y_i)^{c_i} (Q - (1-Q)Y_i)^{c_i+1} \left(\frac{Q - (S-Q)Y_i}{Q - (1-Q)Y_i} \right) \right) \right) \\ & \times \sum_{0 < c_{m+1} < \dots < c_n} \left(\prod_{i=m+1}^n Y_i^{c_i-1} (1+Y_i)^{c_i+1-3} (Q+Y_i)^{-c_i+1} \left(\frac{Q+TY_i}{Q+Y_i} \right) \right) \\ & \quad \times \prod_{1 \leq i < j \leq n} (Q - (1-Q)Y_i + Y_j + Y_i Y_j) (Y_j - Y_i)^{-1}, \end{aligned}$$

which is equal to...

$$\begin{aligned}
& \text{Sym}_Y \left(\prod_{i=1}^m \frac{1}{1+Y_i} \left(\frac{-Y_i}{(1+Y_i)(Q-(1-Q)Y_i)} \right)^{m-i} \right. \\
& \quad \times \left. \left(1 - \prod_{j=1}^i \left(\frac{-Y_j}{(1+Y_j)(Q-(1-Q)Y_j)} \right) \right)^{-1} \frac{Q-(S-Q)Y_i}{Q-(1-Q)Y_i} \right. \\
& \quad \times \prod_{i=m+1}^n (1+Y_i)^{l-2} \left(\frac{Y_i(1+Y_i)}{Q+Y_i} \right)^{i-m-1} \left(1 - \prod_{j=i}^n \left(\frac{Y_j(1+Y_j)}{Q+Y_j} \right) \right)^{-1} \frac{Q+TY_i}{Q+Y_i} \\
& \quad \left. \times \prod_{1 \leq i < j \leq n} (Q-(1-Q)Y_i + Y_j + Y_i Y_j) (Y_j - Y_i)^{-1} \right),
\end{aligned}$$

by using the following (geometric series) evaluation:

$$\sum_{0 \leq c_1 < \dots < c_m} Y_1^{c_1} \dots Y_m^{c_m} = \prod_{i=1}^m Y_i^{i-1} \left(1 - \prod_{j=i}^m Y_j \right)^{-1}.$$

The **Subsets** operator

We define the *subsets* operator as

$$\mathbf{Subsets}_{X_1, \dots, X_m}^{X_{m+1}, \dots, X_n} f(X_1, \dots, X_n) := \sum_{\substack{\sigma \in \mathfrak{S}_n, \\ \sigma(1) < \dots < \sigma(m), \\ \sigma(m+1) < \dots < \sigma(n)}} f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

It follows that

$$\begin{aligned} & \mathbf{Sym}_{X_1, \dots, X_n} f(X_1, \dots, X_n) \\ &= \mathbf{Subsets}_{X_1, \dots, X_m}^{X_{m+1}, \dots, X_n} \mathbf{Sym}_{X_1, \dots, X_m} \mathbf{Sym}_{X_{m+1}, \dots, X_n} f(X_1, \dots, X_n). \end{aligned}$$

Antisymmetriser lemma

We can compute the (anti)symmetriser!

Lemma (H.)

Let $m \geq 1$, then

$$\begin{aligned} \mathbf{ASym}_{X_1, \dots, X_m} & \left(\prod_{r=1}^m \frac{\left(\frac{X_r(1+X_r)}{Q+X_r} \right)^{r-1}}{1 - \prod_{j=r}^m \frac{X_j(1+X_j)}{Q+X_j}} \prod_{1 \leq s < t \leq m} (Q + (Q-1)X_s + X_t + X_s X_t) \right) \\ & = \prod_{r=1}^m \frac{Q + X_r}{Q - X_r^2} \prod_{1 \leq s < t \leq m} \frac{(Q(1 + X_s)(1 + X_t) - X_s X_t)(X_t - X_s)}{Q - X_s X_t}. \end{aligned}$$

We obtain

$$\text{Subsets}_{Y_1, \dots, Y_m}^{Y_{m+1}, \dots, Y_n} \left(\prod_{i=1}^m \frac{Q - (S - Q)Y_i}{Q(1 + Y_i)^2 - Y_i^2} \prod_{1 \leq i < j \leq m} \frac{Q - Y_i Y_j}{Q(1 + Y_i)(1 + Y_j) - Y_i Y_j} \right. \\ \times \prod_{i=m+1}^n (1 + Y_i)^{l-2} \frac{Q + TY_i}{Q - Y_i^2} \prod_{m+1 \leq i < j \leq n} \frac{Q(1 + Y_i)(1 + Y_j) - Y_i Y_j}{Q - Y_i Y_j} \\ \left. \times \prod_{i=1}^m \prod_{j=m+1}^n \frac{Q - (1 - Q)Y_i + Y_j + Y_i Y_j}{Y_j - Y_i} \right).$$

Using the Cauchy determinant

$$\det_{1 \leq i, j \leq n} \left(\frac{1}{X_i + Y_j} \right) = \frac{\prod_{1 \leq i < j \leq n} (X_j - X_i) (Y_j - Y_i)}{\prod_{i, j=1}^n (X_i + Y_j)},$$

this is equal to

$$\text{Subsets}_{Y_1, \dots, Y_m}^{Y_{m+1}, \dots, Y_n} \left(\frac{Q^{\binom{n}{2}}}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2} \det_{1 \leq i, j \leq n} \left(\begin{cases} \frac{Q - (S - Q)Y_i}{Q(1 + Y_i)(1 + Y_j) - Y_i Y_j}, & 1 \leq i \leq m \\ (1 + Y_i)^{l-2} \frac{Q + TY_i}{Q - Y_i^2}, & m + 1 \leq i \leq n \end{cases} \right) \right).$$

The application of **Subsets** Y_{m+1}, \dots, Y_n and the summation over all $1 \leq m \leq n$ yield

$$\frac{Q^{\binom{n}{2}}}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2} \det_{1 \leq i, j \leq n} \left(R \frac{Q - (S - Q)Y_i}{Q(1 + Y_i)(1 + Y_j) - Y_i Y_j} + (1 + Y_i)^{l-2} \frac{Q + TY_i}{Q - Y_i Y_j} \right).$$

Some final manipulation yields

Theorem (H.)

The generating function of (n, l) -ASTs is given by

$$\det_{0 \leq i, j \leq n-1} \left(R \sum_{k=0}^i T^{i-k} \sum_{m=0}^j \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right).$$

But what about the CSSPPs?

The application of **Subsets** Y_{m+1}, \dots, Y_n and the summation over all $1 \leq m \leq n$ yield

$$\frac{Q^{\binom{n}{2}}}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)^2} \det_{1 \leq i, j \leq n} \left(R \frac{Q - (S - Q)Y_i}{Q(1 + Y_i)(1 + Y_j) - Y_i Y_j} + (1 + Y_i)^{l-2} \frac{Q + TY_i}{Q - Y_i Y_j} \right).$$

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But what about the CSSPPs?

Andrews' determinant

If we set $Q = R = S = T = 1$,

$$\det_{0 \leq i, j \leq n-1} \left(R \sum_{k=0}^i T^{i-k} \sum_{m=0}^j \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right)$$

simplifies to

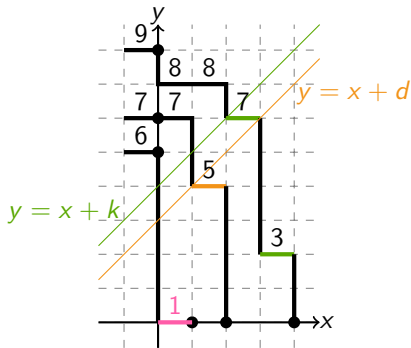
$$\det_{0 \leq i, j \leq n-1} \left(\binom{i+j+l-1}{i} + \delta_{i,j} \right),$$

which enumerates descending plane partitions with no part larger than $n-1$ (Andrews 1979). This can be proved by a nonintersecting lattice paths approach and the Lindström-Gessel-Viennot lemma!

CSSPPs and nonintersecting lattices paths

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class $k=4$



For a fixed parameter $d \in \{1, \dots, k\}$:

- R -weight: # paths
- S -weight: # horizontal steps below the line $y = x + d$ and touching it
- T -weight: # horizontal steps at height 0
- Q -weight: # horizontal steps below the line $y = x + k$ not already counted

Final result

By the Lindström-Gessel-Viennot lemma, we obtain as generating function

$$\sum_{r=0}^n R^r \sum_{0 \leq u_1 < \dots < u_r \leq n-1} \det_{0 \leq i, j \leq r} \left(\sum_{k=0}^{u_i} T^{u_i-k} \sum_{m=0}^{u_j} \binom{u_j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) \right).$$

Theorem (H.)

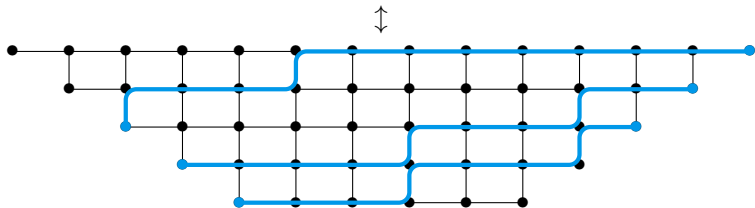
The generating function of (n, l) -ASTs and of CSSPPs of class $l-1$ with no row length larger than n coincide and is given by

$$\det_{0 \leq i, j \leq n-1} \left(R \sum_{k=0}^i T^{i-k} \sum_{m=0}^j \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right).$$

Is there a bijective proof?

Osculating paths: lattice paths that neither cross nor share edges but potentially share points.

0	0	0	0	0	1	0	0	0	0	0	0	0	0
	0	1	0	0	-1	0	0	0	0	1	0	0	
		0	0	0	0	0	1	0	0	0	0		
			1	0	0	0	0	0	0	0			
				1	0	0	-1	0	1				



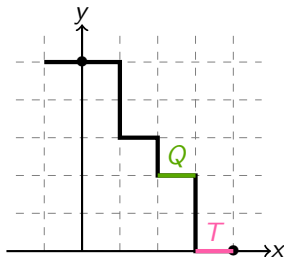
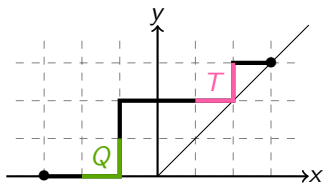
Problem

Find a general (Lindström-Gessel-Viennot type of) enumeration formula for osculating paths!

Weight-preserving bijection for $r = 1$

```
0 0 0 0 0 0 0 0 1 0
  0 0 0 0 1 0 0 0 0
    0 0 0 0 0 1
      1 0 -1 1
        0 1
```

6	6	4	3	1
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Question

Is there a generalisation for $r \geq 2$?

Thank you for your attention!